On the Linearity of Certain Coordinate Transformations Including the Lorentz Transformation

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It is shown that, given choices of units and of synchronization in the moving systems which are independent of the time and space coordinates, the linearity of the transformation equations connecting uniformly moving reference systems is a consequence of the fact that the functions deciding the behaviour of moving bodies and clocks are independent of the time and space coordinates ("homogeneity of time and space").

Introduction

The problem of the linearity of the Lorentz transformations has been discussed by many authors [1]. Different starting points have been chosen, as e.g. the wave equation, Newton's first law, homogeneity of inertial frames, etc. The principle of relativity is often used at some stage at these derivations. Here, continuing in the tradition of FitzGerald, Lorentz, Larmor, Poincaré, Ives, Podlaha, and others, the principle of relativity will not be used, but a set of very general linear transformations will be derived, comprising the Galilei and Lorentz transformations as special cases. It will be shown that, presupposing a convenient choice of units and of synchronization, their linearity is a consequence of the fact that the functions deciding the behaviour of moving bodies and clocks are independent of the time and space coordinates ("homogeneity of time and space"). This derivation will be somewhat different from most other derivations in that it explicitely separates the made *physical assumptions* about the homogeneity of time and space and the (conventional) *choices* leading to the transformation. Besides this separation of conventions and physical assumptions, the main advantage of the present deduction is probably that no recourse is made to mechanics. Let us also note that the homogeneity of time and space, which is decisive for the derivation, is here expressed by utilizing quantities within one reference system only without invoking the concept of transformation, contrary to what is usually the case.

The present paper may be seen as a natural complement to the author's recent note in this Journal [2] where the functions describing the behaviour of moving bodies and clocks were derived from a unifying principle ("Poincaré's principle").

1. Assumptions about the Physical World

In order to simplify the discussion, we shall only consider a two-dimensional space. It is easily seen that this means no essential loss of generality.

Let us assume that time and space are homogeneous and that space is isotropic *. This assumption is here interpreted as follows: there exists a privileged reference frame F_0 with respect to which the description of the world is homogeneous and isotropic when the unit choice is standard and the synchronization absolute [2, 3]. The choice of units being "standard" means that it is independent of the position in time and space and independent of the direction in space. As a consequence of the assumed homogeneity of time and space and isotropy of space, the absolute, the "standard" [3], the "slow clock" [3], and the "rapid clock" [4] synchronizations all coincide. Note that this coincidence holds only in the privileged frame F_0.

Furthermore, we assume quite generally that the dimensions of bodies moving with respect to F_0 are changed: in the direction of motion by a factor Φ, in the transversal direction by a factor Ψ; and that the rates of moving clocks are changed by a factor

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* The assumption about isotropy of space is not necessary for linearity. The derivation may easily be generalized to the non-isotropic case.

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Ω [1]. In accord with the character of the privileged frame F_0, we assume that these functions are independent of the time and space coordinates and are direction independent. In general they could depend on velocity and acceleration, however. Let us to this end make the following very general assumption: the dimensions of bodies and the rates of clocks depend only on their actual state of motion, i.e. the dimensions of a body and the rate of a clock at a certain time depend only on their velocity and acceleration at that time and not on past or future velocities or accelerations. Since we shall here never consider accelerating objects, i.e. measurements will only be undertaken on objects at rest or in uniform motion, this assumption allows us to omit the acceleration dependence and write the functions as follows:

$$\Phi = \Phi(|\boldsymbol{w}|), \quad \Psi = \Psi(|\boldsymbol{w}|), \quad \Omega = \Omega(|\boldsymbol{w}|).$$

They are assumed to be continuous functions of $\boldsymbol{w}$, the velocity with respect to F_0 of the moving object.

2. Derivation of the General Transformation

We shall now derive a coordinate transformation between a Cartesian reference system S_0 in F_0 and another Cartesian system S' in a frame F' moving at the constant velocity u with respect to F_0. The physics needed for this task is contained in the previous section. What remains is a number of choices.

Let us denote by x, y, t the coordinates of an event in S_0 and by x', y', t' the coordinates of the same event in S'. We may then write the general transformation

$$x' = f(x, y, t, u), \quad y' = g(x, y, t, u), \quad t' = h(x, y, t, u), \tag{1a}$$

with the inverse

$$x = f'(x', y', t', u), \quad y = g'(x', y', t', u), \quad t = h'(x', y', t', u). \tag{1b}$$

Note that the inverse must exist since coordinates are by definition one-to-one.

We divide the choices into three groups:

Group I. Choice of origin, orientation, and direction of motion of S'.

a) We choose the origins of S_0 and S' to coincide at $t' = t = 0$:

$$f(0,0,0,u) = f'(0,0,0,u) = g(0,0,0,u) = g'(0,0,0,u) = 0, \quad h(0,0,0,u) = h'(0,0,0,u) = 0. \tag{2}$$

b) We choose, at the time $t = 0$, the y'-axis parallel to the y-axis and the x'-axis parallel to the x-axis:

$$f(x_0, 0, 0, u) = 0 \Rightarrow f(x_0, y, 0, u) = 0 \quad \text{for all } y, \tag{3a}$$

$$g(0, y_0, 0, u) = 0 \Rightarrow g(x, y_0, 0, u) = 0 \quad \text{for all } x. \tag{3b}$$

c) We choose the direction of motion of S' to be along the x-axis of S_0 and decide that S' will not rotate with respect to S_0. This means demanding that the axises of S' are both moving at the velocity u in the x-direction:

$$f(x_0, y_0, 0, u) = 0 \Rightarrow f(x_0 + ut, y_0, t, u) = 0 \quad \text{for all } t, \tag{4a}$$

$$g(x_0, y_0, 0, u) = 0 \Rightarrow g(x_0 + ut, y_0, t, u) = 0 \quad \text{for all } t. \tag{4b}$$

Let us note that from (2), (3a), and (4a) follows

$$f(ut, y, t, u) = 0, \tag{5a}$$

and that from (2), (3b), and (4b) follows

$$g(x, 0, t, u) = 0. \tag{5b}$$

In order not to complicate our formulae unnecessarily, we shall in the following at the mathematical formulation of the choices to be made presuppose (3a, b) and (4a, b), i.e. that the x'-axis and the

[1] This notation was introduced by Podlaha [5]. To be complete one should make an assumption about the change of dimension in any given direction. Cf. the next footnote.
The form of the functions Φ, Ψ, and Ω will of course depend on the method of their measurement, as e.g. the chosen synchronization in F_0. In this sense they are conventional, too. Once decided this, however, one is able to say something new about the physical world. Contrary to this, the choices to be made in Sect. 2 do not enable us to say anything new about the physical world. Note also that the physically important statement about the isotropy of space and the homogeneity of time and space is not that we actually choose the units and synchronization in the above-mentioned way, but that if we do so, then we obtain isotropy and homogeneity.

y'-axis are parallel to the x-axis and y-axis respectively and that S' is moving at the velocity u in the direction of the x-axis.

d) We choose the positive directions on the axises in the same way:

$$f(x_2, y, t, u) > f(x_1, y, t, u) \quad \text{iff} \quad x_2 > x_1, \tag{6a}$$

$$g(x, y_2, t, u) > g(x, y_1, t, u) \quad \text{iff} \quad y_2 > y_1, \tag{6b}$$

$$h'(x', y', t_2', u) > h'(x', y', t_1', u) \quad \text{iff} \quad t_2' > t_1'. \tag{6c}$$

Group II. Choice of units in S'.

We choose units in the usual way elsewhere called "preserving matter-geometry" [3]; i.e. we stipulate that a clock which at rest in S_0 has the frequency ν_0, will, after being accelerated and brought to rest in S', have the frequency ν_0 as measured in S'. Likewise for measuring-rods[2]. Exploiting our knowledge about the dimensions of moving bodies and the rates of moving clocks (Sect. 1), this gives us three relations between the coordinates in S_0 and S':

a) The choice of unit in the direction of the x'-axis is equivalent to

$$|x_2 - x_1| = |f(x_2, y, t, u) - f(x_1, y, t, u)|\, \Phi(u). \tag{7a}$$

b) The choice of unit in the direction of the y'-axis is equivalent to

$$|y_2 - y_1| = |g(x, y_2, t, u) - g(x, y_1, t, u)|\, \Psi(u). \tag{7b}$$

c) The choice of the unit of time is equivalent to

$$|t_2' - t_1'| = |h'(x', y', t_2', u) - h'(x', y', t_1', u)|\, \Omega(u). \tag{7c}$$

Let us, before introducing the third group of choices-relations, see what may be deduced from that we have already got:

[2] Let us note that with this choice of units Pythagoras' law is in general not valid in S'. Denoting by $\Phi_\theta(u)$ the function describing the change of dimensions of a body in the direction θ with the direction of motion ($\Phi_0(u) = \Phi(u)$, $\Phi_{\pi/2}(u) = \Psi(u)$), the condition for its validity is the following: $\Phi_\theta(u) = (\Psi^{-2} \sin^2\theta + \Phi^{-2}\cos^2\theta)^{-1/2}$. This condition is fulfilled in the Newtonian as well as in the Lorentzian world.

i) From (7a), (6a), and (5a) we obtain:

$$x - ut = [f(x, y, t, u) - f(ut, y, t, u)]\, \Phi(u) = f(x, y, t, u)\, \Phi(u),$$

i.e.

$$x' = f(x, y, t, u) = \Phi(u)^{-1}(x - ut). \tag{8a}$$

ii) From (7b), (6b), and (5b) we obtain:

$$y - 0 = [g(x, y, t, u) - g(x, 0, t, u)]\, \Psi(u) = g(x, y, t, u)\, \Psi(u),$$

i.e.

$$y' = g(x, y, t, u) = \Psi(u)^{-1} y. \tag{8b}$$

iii) From (7c) and (6c) we obtain

$$t' - 0 = [h'(x', y', t', u) - h'(x', y', 0, u)] \cdot \Omega(u),$$

i.e.

$$t = h'(x', y', t', u) = \Omega(u)^{-1} t' - \Omega(u)^{-1} t_1' + h'(x', y', t_1', u). \tag{8c}$$

We have thus already found the form of the first two transformation equations. Their linearity emerged in a straight-forward way from the fact that the functions Φ and Ψ are independent of the time and space coordinates. We have also found that the third inverse transformation equation is linear in t'.

Group III. Choice of synchronization in S'.

Choosing a synchronization in S' means deciding which are to be the differences in readings between the clocks in different points at some arbitrary but fixed time, as e.g. $t = 0$. Every physical phenomenon may be described with in principle any synchronization; the most extreme possibility would be to assign the time values to the different clocks completely arbitrarily, without any system. However, there exist more practical methods of which we choose the following: the difference in readings at the time $t = 0$ between two clocks is only to depend on the difference between their respective x'-coordinates and on the difference between their respective y'-coordinates. We also demand that the difference between two infinitely adjacent clocks is infinitesimal. This prescription may be expressed in the following way:

$$h'(0, 0, t_0', u) = 0 \Rightarrow h'(x', y', t_0' + A(u)\,\Phi(u)\,x' + B(u)\,\Psi(u)\,y', u) = 0 \quad \text{for all } x' \text{ and } y', \tag{9}$$

where A and B are some arbitrary functions of u.

Combining (9) and (2), we obtain:

$$h'(x', y', A(u)\,\Phi(u)\,x' + B(u)\,\Psi(u)\,y', u) = 0$$
$$\text{for all } x' \text{ and } y'. \tag{10}$$

To obtain the third inverse transformation equation, it is now sufficient to choose

$$t_1' = A(u)\,\Phi(u)\,x' + B(u)\,\Psi(u)\,y'$$

in (8c) and to make use of the relation (10). We write down the whole transformation and its inverse:

$$\begin{aligned} x' &= \Phi^{-1}(x - u\,t), \\ y' &= \Psi^{-1} y, \\ t' &= \Omega\,t + A(x - u\,t) + B\,y, \end{aligned} \tag{11a}$$

$$\begin{aligned} x &= \Phi x' + u\,\Omega^{-1} t' - u\,\Omega^{-1}(A\,\Phi x' + B\Psi y'), \\ y &= \Psi y', \\ t &= \Omega^{-1} t' - \Omega^{-1}(A\,\Phi x' + B\Psi y'). \end{aligned} \tag{11b}$$

The coefficients Φ, Ψ, Ω, A, and B are to be evaluated at u.

This is a slightly more general transformation than the one derived in [3], where the synchronization in the transversal direction was chosen the same in S′ as in S_0, which amounts to setting $B \equiv 0$. We remark that the choice $B \neq 0$ is not very interesting in an isotropic world. In a Newtonian world this holds true: $\Phi = \Psi = \Omega \equiv 1$. The Galilei transformation is then obtained by setting $A = B \equiv 0$. In a Lorentzian world we have:

$$\Phi = \Omega = \sqrt{1 - u^2/c^2} \quad \text{and} \quad \Psi \equiv 1\,.$$

The Lorentz transformation is then obtained by setting $B \equiv 0$ and $A = \Omega'$.

3. Discussion

Let us reflect upon the made choices and why the transformation turned out linear.

The first group of choices is not very substantial. We could, of course, choose different origins and orientations and let S′ move in an arbitrary direction (at constant velocity), and the equations would still be linear, only they would have a more complicated form. The most general transformation is obtained from (11a, b) by application of translations, rotations, and inversions.

Besides some condition about the uniform motion of S′ with respect to S_0 of type (4a, b), the linearity is hence a consequence of the choices of units (group II), of synchronization (group III), and especially of the fact that Φ, Ψ, and Ω do not depend on the space or time coordinates (Section 1). Let us point out that even given the choice of synchronization and the form of the matter functions Φ, Ψ, and Ω, there is a whole group of unit choices implying linearity: all time and space independent choices are acceptable, i.e. we could, instead of letting the accelerated clock have the same frequency ν_0 at rest in S′ as before at rest in S_0, decide it to have the frequency $H(u)\,\nu_0$, where H is a function of u only. Likewise we could introduce arbitrary functions F and G for the choice of length units. In the transformation equations (11a, b) we then would set $F^{-1}\Phi$ instead of Φ, $G^{-1}\Psi$ instead of Ψ, and $H^{-1}\Omega$ instead of Ω. This choice of units in S′ is, however, rather artificial and unpractical. In particular for $G \neq F$ a rod turned from the x'-axis to the y'-axis would be said to have changed its length. We note that the choice $F = \Phi$, $G = \Psi$, and $H = \Omega$ together with a suitable synchronization yields the Galilei transformation[3]. Notwithstanding the arbitrariness and practical inconveniences of not choosing all three functions F, G, and H identical one, I think it important to be aware of the fact that it is possible to describe every physical phenomenon with any choice of units. This is intimately connected with Ives' conclusion [7] "that the variations of mass, length, and frequency with motion are the primary physical phenomena, the Lorentz transformations are consequences" and with Podlaha's requirement that a good physical theory must be able to solve every physical problem in just *one* inertial frame [5]. We may summarize this in saying that physical reality is represented by the functions Φ, Ψ, and Ω, while the Lorentz transformations serve as a means of translating the descriptions by the various inertial observers.

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[3] If Poincaré's principle about the impossibility to detect absolute motion is valid, the realization of this choice is impossible, however (cf. [2]).

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